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Optimal Relocation of Satellites Flying in Near-Circular-Orbit Formations

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In this paper we present a general analytic formulation for optimal transfer paths for satellites flying in formation based upon the circular Hill's problem. Optimization is performed to minimize the transfer energy required from the thruster. We consider the optimization problem as the choice of trajectory the satellite should follow during the maneuver, whereas the time taken is fixed as are the boundary conditions. We show that this optimal control problem has simple analytic solutions that provide a powerful basis from which to develop formation control strategies. In formation flying, maneuvers require low levels of thrust, and we assume that the thruster will be firing throughout the maneuver and the optimization scheme solves for the magnitude and direction of thrust as functions of time. We illustrate how we can exploit these analytic results and present examples of maneuvers using them. We also present a discussion of a docking problem, where we reverse the analysis and solve for initial conditions fixing the thruster characteristics in the optimal solutions found. Finally, we present a discussion of how we might exploit the natural dynamics to gain further propellant savings by adjusting the boundary conditions. This is illustrated by considering a cross-track maneuver for plane change.

Introduction

THERE is great interest in missions involving groupings of satellites flying in formation. These groupings require orbital control systems to maintain the formation. The description of the relative motions of satellites flying in close proximity is provided by the Hill's equations,[1] and when thrusters are used the inhomogeneous equations need to be solved.

It is common practice, when considering orbital maneuvers, to assume that the thrust is provided instantaneously, and can then be expressed in the form of a Δv. In practice, however, the thrust levels are usually quite small, and significantly long burns are required to make changes to the relative orbits of the satellites. There is also increased interest in systems with very low thrust levels that are active continuously.[2]

The maneuver of a satellite in a local circular Hill frame by a continuous constant thrust that can be rotated was considered by Hinz[3] and others.[4] This work was then extended to include variable thrust magnitude as well as direction by Gobetz[5] and maneuvers between elliptic orbits by Marinescu.[6] The effects of incorporating mass loss as propellant that is expelled have been formalized by Prussing.[7] More recently this question has been considered again, this time in the context of optimizing either time of transfer[8] or fuel optimization.[9,10] Analytic solutions incorporating perturbing forces have been presented in Refs. 11 and 12. The reason for this renewed interest is the development of formation flying as a paradigm for future space missions and the development of the International Space Station.[13–15] This has led to some demonstration missions as described in Ref. 16 for automated rendezvous and docking. The use of small satellites in formation requires the development of propellant optimal maneuvers, critical for extending the lifetime of such missions.[17]

The control of such formations and proximity operations has also attracted considerable attention. The use of optimal control techniques has been widely exploited for stationkeeping.[18–20] All of these papers require the numerical solution of a Ricatti equation to minimize the deviation of the location of one satellite with respect to another. The problem of maneuvering a satellite within a formation from one relative orbit to another has been considered by Starin et al.,[21] who used an optimal controller for a leader-follower formation. This again requires a numerical solution, and control is only applied in two directions. Other approaches to optimization, which take account of various constraints, such as thruster limitations or restrictions on path require nonlinear programming.[22]

There is some value in returning to the question of minimum propellant transfers for satellites flying in formation and to seek analytic solutions. Such solutions, if found, would be very helpful in devising feedback controllers that are not as dependent upon numerical solutions of the control equations and could provide suitable novel controllers for formation flying.

In this paper we are concerned with investigating orbital maneuvers for which the burn time of the thruster is a significant fraction of an orbital period. The thrusters are assumed to be able to provide a variable thrust acceleration, and the satellite is assumed to have the capability of providing three arbitrary thrust accelerations in orthogonal directions. This will require a level of attitude control during the maneuver as well as three thrusters.

We restrict ourselves to formations of satellites that are in nearly circular orbits so that we can employ the standard Hill's equations.[1] We shall review these equations and provide a formal solution to them for a variable, unspecified thrust vector as a function of time. We will go on to discuss the optimization problem that we shall address involving boundary conditions at the start and end of a fixed time interval. We shall show in this paper that there exist simple analytic forms for the three thrust accelerations as functions of time which minimize the propellant usage and illustrate how these analytic results can be used for formation maintenance and maneuvers. Our results are not directly comparable to optimal control solutions, as we seek the path by which a satellite in the formation will maneuver. Feedback control will be required to mitigate against disturbances to the dynamics and errors in the thrust accelerations delivered during the maneuver.

Solution to Hill's Equations

Homogeneous Equations

We shall consider maneuvers of satellites in nearly circular orbit and so we can describe the motion in terms of a circular guiding centre orbit plus an epicycle. This solution is derived from the homogeneous Hill's equations:

$$\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x = 0, \qquad \ddot{y} + 2\Omega\dot{x} = 0, \qquad \ddot{z} + \Omega^2 z = 0 \quad (1)$$

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where the x axis lies in the radial direction, the y axis is along the in-track direction of the guiding center, and the z axis is in the cross-track direction. Ω is the angular rotation rate of the frame of reference, which rotates at the same rate as the guiding center. One can easily solve these equations for arbitrary initial conditions. The solutions can be expressed as follows[23]:

$$\begin{aligned} x &= b + A\cos\alpha & \dot{x} &= -A\Omega\sin\alpha \\ y &= w - \tfrac{3}{2}b\alpha - 2A\sin\alpha & \dot{y} &= -\tfrac{3}{2}b\Omega - 2A\Omega\cos\alpha \\ z &= Z\cos(\alpha-\alpha_N) & \dot{z} &= -Z\Omega\sin(\alpha-\alpha_N) \end{aligned} \tag{2}$$

where $\alpha = \Omega(t - t_A)$ and is called the epicycle phase; A is the epicycle amplitude; b is a radial offset ($b > 0$ implies further from the gravitational mass center); Z is the amplitude of the cross-track excursion; w the along-track offset; t_A the time of apocenter passage; and $\alpha_N = \Omega(t - t_N)$, where t_N is the time of nodal passage. The guiding center radius is a, which is related to the rotation rate Ω through the usual Kepler relation: $\Omega^2 a^3 = GM$, where G is the gravitational constant and M the gravitational mass about which the satellites orbit. The semimajor axis is approximately $a + b$, and the eccentricity is A/a. The six orbital elements in this representation are (b, A, w, t_A, Z, t_N).

It is sometimes useful to re-express this propagation of Hill's coordinates through a state transition matrix (for example, see Ref. 12). When the epicycle phase has changed by an angle β after a time t, then

$$\begin{bmatrix}\Omega\hat{x}\\ \hat{\dot{x}}\\ \Omega\hat{y}\\ \hat{\dot{y}}\end{bmatrix} = \begin{bmatrix} 4-3\cos\beta & \sin\beta & 0 & 2(1-\cos\beta)\\ 3\sin\beta & \cos\beta & 0 & 2\sin\beta\\ -6(\beta-\sin\beta) & -2(1-\cos\beta) & 1 & -(3\beta-4\sin\beta)\\ -6(1-\cos\beta) & -2\sin\beta & 0 & -(3-4\cos\beta)\end{bmatrix} \times \begin{bmatrix}\Omega x_0\\ \dot{x}_0\\ \Omega y_0\\ \dot{y}_0\end{bmatrix}$$

We have separated off the z motion from this state transition matrix as this motion is decoupled from the in-plane motion. Hence for the out-of-plane motion,

$$\begin{bmatrix}\Omega\hat{z}\\ \hat{\dot{z}}\end{bmatrix} = \begin{bmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\Omega z_0\\ \dot{z}_0\end{bmatrix}$$

We have used the caret to represent the evolution of position and velocity in the absence of any control accelerations. We now consider using a continuous low level of thrust to maneuver the satellite from one orbit to another, changing these epicycle parameters.

Inhomogeneous Equations

We suppose the satellite has three sets of thrusters pointing in the directions of x, y, and z. The equations of motion when these thrusters are firing are given by

$$\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x = T_x, \qquad \ddot{y} + 2\Omega\dot{x} = T_y, \qquad \ddot{z} + \Omega^2 z = T_z \tag{3}$$

where T_x, T_y, and T_z are the thrust accelerations and are to be considered arbitrary functions of time. As the motion out of plane is completely decoupled from the in-plane motion, we can consider this separately. One can solve the inhomogeneous equations by employing the solutions of the homogeneous equations using the method of variation of parameters.[24]

Out-of-Plane Maneuvers

Without loss of generality, we shall assume that the z thruster is switched on at time $t = 0$. At this time the position and velocity out of plane of the satellite is $(z_0, \dot{z}_0)$. The thruster fires continuously with a time-varying thrust amplitude, and at a later time t the position and velocity in the z direction will be given by

$$\begin{bmatrix}\Omega z\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\Omega\hat{z}\\ \hat{\dot{z}}\end{bmatrix} + \begin{bmatrix}I_0\\ I_1\end{bmatrix}$$

where I_0 and I_1 are integral quantities related to the thrust function applied[24]:

$$I_0(t) = \int_0^t T_z(\tau)\sin\Omega(t-\tau)\,\mathrm{d}\tau \tag{4}$$

and

$$I_1(t) = \int_0^t T_z(\tau)\cos\Omega(t-\tau)\,\mathrm{d}\tau \tag{5}$$

After some time t_F the thruster is switched off, and the orbit has changed to some predefined target orbit. In the cross-track direction this is signified by $(\Omega z_F, \dot{z}_F)$. To guarantee that the satellite has reached the target location in time t_F, the preceding two integrals provide constraints upon the thrust function. Let $I_0(t_F) = \mathfrak{I}_0$ and $I_1(t_F) = \mathfrak{I}_1$, then

$$\begin{bmatrix}\mathfrak{I}_0\\ \mathfrak{I}_1\end{bmatrix} = \begin{bmatrix}\Omega z_F\\ \dot{z}_F\end{bmatrix} - \begin{bmatrix}\Omega\hat{z}_F\\ \hat{\dot{z}}_F\end{bmatrix}$$

where $\hat{z}_F$ and $\hat{\dot{z}}_F$ are the position and velocity the satellite would have at time t_F if the thruster were not fired. We should think of this equation as defining constraints upon T_z caused by the required boundary conditions at the start and end of the maneuver. The first term on the right is the predefined target state, and the second term is just a function of how long we are prepared to fire to get to the target.

In-Plane Maneuvers

The analysis of the cross-track motion can be carried through in a straightforward manner to the in-plane motion. In this case, however, there are two coupled equations and two thrust functions T_x in the radial direction and T_y along track. Firing these two thrusters for a time t changes the phase space location in the in-track direction by

$$\begin{bmatrix}\Omega x\\ \dot{x}\\ \Omega y\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\Omega\hat{x}\\ \hat{\dot{x}}\\ \Omega\hat{y}\\ \hat{\dot{y}}\end{bmatrix} + \begin{bmatrix}2 & -2 & 0 & 0\\ 0 & 0 & 2 & 0\\ -3(\Omega t) & 0 & 4 & 3\\ -3 & 4 & 0 & 0\end{bmatrix}\begin{bmatrix}I_2\\ I_3\\ I_4\\ I_5\end{bmatrix}$$

where[24]

$$\begin{aligned} I_2(t) &= \int_0^t T_y(\tau)\,\mathrm{d}\tau \\ I_3(t) &= \int_0^t T_y(\tau)\cos\Omega(t-\tau)\,\mathrm{d}\tau - \frac{1}{2}\int_0^t T_x(\tau)\sin\Omega(t-\tau)\,\mathrm{d}\tau \\ I_4(t) &= \int_0^t T_y(\tau)\sin\Omega(t-\tau)\,\mathrm{d}\tau + \frac{1}{2}\int_0^t T_x(\tau)\cos\Omega(t-\tau)\,\mathrm{d}\tau \\ I_5(t) &= \Omega\int_0^t T_y(\tau)\tau\,\mathrm{d}\tau - \frac{2}{3}\int_0^t T_x(\tau)\,\mathrm{d}\tau \end{aligned} \tag{6}$$

After firing for a duration t_F, we require the orbit to reach a predefined final location in phase space $(\Omega x_F, \dot{x}_F, \Omega y_F, \dot{y}_F)$, which provides four integral constraints:

$$\begin{bmatrix} \mathcal{I}_2 \\ \mathcal{I}_3 \\ \mathcal{I}_4 \\ \mathcal{I}_5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 3/2 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 0 \\ 2\beta & -2/3 & 1/3 & \beta \end{bmatrix} \begin{bmatrix} \Omega(x_F - \hat{x}_F) \\ \dot{x}_F - \hat{\dot{x}}_F \\ \Omega(y_F - \hat{y}_F) \\ \dot{y}_F - \hat{\dot{y}}_F \end{bmatrix}$$

This completes the derivation of the integral constraints to transfer between and initial and final orbit. The expressions on the left-hand side are integrals over the thrust accelerations, which have to satisfy the preceding conditions for the orbital boundary conditions to be met.

Optimization Problem Statement

In this section we shall consider how to choose the thrust accelerations in an optimal way while preserving the constraints defined in the preceding section. There are two main considerations when considering the thrust: the first is to minimize the amount of propellant required, and the second is to minimize the time required for the maneuver. For a small satellite the use of propellant has the highest priority because of the small capacity for propellant storage. Nevertheless for formation-flying applications, the reorientation of the formation within a set time is essential for a flexible system, maximizing data collection and collision-avoidance planning.

To minimize the propellant usage, we seek to minimize the actuation required for the maneuver. This also minimizes the propellant usage for a thrust engine operating at constant power. The cost function to minimize control actuation is

$$\mathcal{J} = \int_0^{t_F} T(\tau)^2 \, \mathrm{d}\tau \tag{7}$$

where the thrusters continuously accelerate the spacecraft over a fixed time interval t_F. This is dictated by the formation planning system before the maneuver. The problem then is to minimize the preceding cost function subject to boundary constraints on position and velocity at the start and end of the fixed time interval t_F.

The problem we are addressing here is how to determine a reference trajectory that will steer satellites in the formation from their present relative position and velocity to a new desired position and velocity within a fixed time. We are solving for the thrust accelerations in all three directions as functions of time, such that we minimize propellant usage throughout the maneuver.

The relative motion equations decouple the motion out of the orbital plane from the motion within the plane. We can exploit this in our optimality conditions by replacing $\mathcal{J}$ by two positive-definite components $\mathcal{J}_z$ and $\mathcal{J}_{xy}$, where

$$\mathcal{J}_z = \int_0^{t_F} T_z(\tau)^2 \, \mathrm{d}\tau \tag{8}$$

and

$$\mathcal{J}_{xy} = \int_0^{t_F} T_x(\tau)^2 + T_y(\tau)^2 \, \mathrm{d}\tau \tag{9}$$

In the next section we shall incorporate the constraints upon this optimization problem that arise from the meeting the boundary conditions of position and velocity at the final time t_F. We shall treat these separately for the cross-track motion and the in-plane motion.

The approach we shall adopt in solving this optimization problem is to represent the thrust accelerations as periodic functions of time with period t_F. Because the thrust accelerations are only active over the time interval $0 \leq t \leq t_F$, then extending them as periodic functions beyond this time interval does not affect the maneuver. Once we allow the thrust accelerations to be periodic, then it is natural to represent them by a Fourier series with period t_F. Note that the period in these Fourier series is different from the orbital period given by $2\pi/\Omega$.

The use of Fourier series to represent the thrust accelerations puts no significant limitation upon these functions that are given by the Dirichlet conditions.[25] The Fourier series is just a description of an arbitrary function in terms of a complete set of orthogonal functions. The thrust accelerations can be discontinuous and even go to infinity a finite number of times.

By using the Fourier representation, we can express the cost function in terms of the unknown Fourier coefficients in the series and solve for these coefficients. The problem then becomes a parametric one, although there are an infinite number of parameters required to represent arbitrary thrust accelerations. We can use Parseval's theorem[26] to express the preceding cost functions in terms of these parameters.

We start by defining the Fourier series of the thrust accelerations as

$$T_x = \frac{a_{x0}}{2} + \sum_{n=1}^{\infty} a_{xn} \cos(n\Omega_F t) + b_{xn} \sin(n\Omega_F t)$$

$$T_y = \frac{a_{y0}}{2} + \sum_{n=1}^{\infty} a_{yn} \cos(n\Omega_F t) + b_{yn} \sin(n\Omega_F t)$$

$$T_z = \frac{a_{z0}}{2} + \sum_{n=1}^{\infty} a_{zn} \cos(n\Omega_F t) + b_{zn} \sin(n\Omega_F t) \tag{10}$$

where $\Omega_F = 2\pi/t_F$. The cost function, therefore, for the z thruster can then be written as

$$\mathcal{J}_z = \frac{t_F}{2}\left[\frac{a_{z0}^2}{2} + \sum_{n=1}^{\infty} a_{zn}^2 + b_{zn}^2\right] \tag{11}$$

whereas in the in-plane direction

$$\mathcal{J}_{xy} = \frac{t_F}{2}\left(\frac{a_{x0}^2}{2} + \sum_{n=1}^{\infty} a_{xn}^2 + b_{xn}^2 + \frac{t_F}{2}\right)\frac{a_{y0}^2}{2} + \sum_{n=1}^{\infty} a_{yn}^2 + b_{yn}^2 \tag{12}$$

Before we embark upon minimizing these functions with respect to the Fourier coefficients, we need to incorporate the boundary constraints. In the next section we will derive the constraints in terms of the Fourier series and incorporate them into the cost functions and then solve for the optimal thrust accelerations.

Cross-Track Thrust Function

We derive here the optimal thrust function for cross-track maneuvers and show that this thrust function has a very simple form irrespective of the boundary conditions for the maneuver. We start by returning to the constraints considered earlier and express them in terms of the Fourier representation. The constraints $\mathcal{I}_0$ and $\mathcal{I}_1$ are more conveniently expressed in terms of the symmetric and anti-symmetric components of $T_z(t)$. For the cross-track maneuvers this is achieved by replacing $\mathcal{I}_0$ and $\mathcal{I}_1$ by new constraints: K_0 and K_1, where

$$\cos\varphi \begin{pmatrix} K_0 \\ K_1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathcal{I}_0 \\ \mathcal{I}_1 \end{pmatrix} \tag{13}$$

the rotation angle $\varphi = (\pi - \beta)/2$ and $\beta = \Omega t_F$. These new constraints can be expressed in terms of integrals over the thrust function:

$$K_0 \sin\frac{\beta}{2} = \int_0^{t_F} T_z(\tau) \cos\left(\frac{\beta}{2} - \Omega\tau\right) \mathrm{d}\tau \tag{14}$$

$$K_1 \sin\frac{\beta}{2} = -\int_0^{t_F} T_z(\tau) \sin\left(\frac{\beta}{2} - \Omega\tau\right) \mathrm{d}\tau \tag{15}$$

They can also be expressed in terms of the boundary conditions at the start and end of the maneuver:

$$K_0 = (\dot{z}_F - \hat{\dot{z}}_F)\cot(\beta/2) + \Omega(z_F - \hat{z}_F) \tag{16}$$

$$K_1 = (\dot{z}_F - \hat{\dot{z}}_F) - \Omega(z_F - \hat{z}_F)\cot(\beta/2) \tag{17}$$

Replacing $T_z(t)$ by its Fourier series, we can integrate Eqs. (14) and (15) term by term to find

$$\Omega K_0 = S_{zA} \tag{18}$$

and

$$K_1 = 2\Omega_F S_{zB} \tag{19}$$

where

$$S_{zA} = a_{z0} + 2\Omega^2 \sum_{n=1}^{\infty} \frac{a_{zn}}{\Omega^2 - n^2\Omega_F^2} \tag{20}$$

and

$$S_{zB} = \sum_{n=1}^{\infty} \frac{nb_{zn}}{\Omega^2 - n^2\Omega_F^2} \tag{21}$$

The advantage of using K_0 and K_1 as constraints is that the first is only a function of the a coefficients and the second a function of the b coefficients of the unknown Fourier series. We next incorporate these constraints into the cost function by introducing Lagrange multipliers:

$$J_z = \frac{\beta}{2\Omega}\left[\frac{a_{z0}^2}{2} + \sum_{n=1}^{\infty} a_{zn}^2 + b_{zn}^2\right] + \lambda_0\left(K_0 - \frac{S_{zA}}{\Omega}\right) + \lambda_1(K_1 - 2\Omega_F S_{zB}) \tag{22}$$

It is straightforward to differentiate this cost function with respect to all of the Fourier coefficients. Here K_0 and K_1 are treated as functions of the boundary conditions through Eqs. (16) and (17). Solving for the Fourier coefficients gives

$$a_{z0} = \frac{2\lambda_0}{\beta} \tag{23}$$

$$a_{zn} = \frac{2\lambda_0}{\beta}\frac{\Omega^2}{\Omega^2 - n^2\Omega_F^2} \tag{24}$$

$$b_{zn} = \frac{2\lambda_1}{\beta}\frac{n\Omega\Omega_F}{\Omega^2 - n^2\Omega_F^2} \tag{25}$$

Substituting these into Eqs. (20) and (21) gives

$$\Omega K_0 = \frac{2\lambda_0}{\beta}\left[1 + 2\Omega^4 \sum_{n=1}^{\infty} \frac{1}{\left(\Omega^2 - n^2\Omega_F^2\right)^2}\right] \tag{26}$$

and

$$\Omega K_1 = \frac{4\lambda_1\Omega\Omega_F^2}{\beta}\sum_{n=1}^{\infty} \frac{n^2}{\left(\Omega^2 - n^2\Omega_F^2\right)^2} \tag{27}$$

These infinite sums can be evaluated into closed form, as shown in the Appendix. We can therefore get exact expressions for the Lagrange multipliers λ_0 and λ_1 in terms of K_0 and K_1 using Eqs. (A9) and (A10), respectively:

$$\lambda_0 = 2\Omega K_0 \frac{1 - \cos\beta}{\beta + \sin\beta}, \qquad \lambda_1 = 2\Omega K_1 \frac{1 - \cos\beta}{\beta - \sin\beta} \tag{28}$$

We can also substitute the Fourier coefficients (23), (24), and (25) into the Fourier series of the thrust function T_z to obtain the required function of time. We leave this expression in terms of the Lagrange multipliers, which are determined from the preceding boundary conditions:

$$T_z(t) = \frac{\lambda_0}{\beta}\left(1 + 2\Omega^2 \sum_{n=1}^{\infty} \frac{\cos n\Omega_F t}{\Omega^2 - n^2\Omega_F^2}\right) + \frac{2\Omega\Omega_F\lambda_1}{\beta}\sum_{n=1}^{\infty} \frac{n \sin n\Omega_F t}{\Omega^2 - n^2\Omega_F^2} \tag{29}$$

These Fourier series can also be summed into a closed-form solution (see Appendix). Using Eqs. (A3) and (A4), we find that the optimal thrust acceleration has the simple form:

$$T_z = \Gamma \cos(\Omega t - \zeta) \tag{30}$$

The amplitude for this optimal thrust function is given by

$$\Gamma = \tfrac{1}{2}|\csc(\beta/2)|\sqrt{\left(\lambda_0^2 + \lambda_1^2\right)} \tag{31}$$

and the phase

$$\zeta = \upsilon + \beta/2 \tag{32}$$

where

$$\lambda_1 = \lambda_0 \tan\upsilon \tag{33}$$

To determine λ_0 and λ_1, we use Eqs. (28), where K_0 and K_1 are given by Eqs. (16) and (17), respectively. The cost function can be evaluated to the simple result:

$$\mathcal{J}_z = \tfrac{1}{2}(K_0\lambda_0 + K_1\lambda_1) \tag{34}$$

Finally, the functions $I_0(t)$ and $I_1(t)$ are

$$I_0(t) = (\Gamma/4\Omega)[\cos(\Omega t - \zeta) - \cos(\Omega t + \zeta)] + (\Gamma/2)\sin(\Omega t - \zeta)t \tag{35}$$

$$I_1(t) = (\Gamma/4\Omega)[\sin(\Omega t - \zeta) + \sin(\Omega t + \zeta)] + (\Gamma/2)\cos(\Omega t - \zeta)t \tag{36}$$

These expressions can then be substituted back to find the cross-track position and velocity of the satellite throughout the maneuver. This completes the solution for the optimal thruster function for cross-track maneuvers.

In-Plane Thrust Functions

The analysis presented for the cross-track thrust acceleration can be extended to the thrust accelerations in the radial and in-track directions. Because the analysis parallels exactly the previous, we shall just provide only the key steps in the argument.

We again modify the constraints (6) to more convenient forms:

$$\begin{pmatrix} K_2 \\ K_3\cos\varphi \\ K_4\cos\varphi \\ K_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 1 & 0 & 0 & -2/\beta \end{pmatrix} \begin{pmatrix} \mathcal{I}_2 \\ \mathcal{I}_3 \\ \mathcal{I}_4 \\ \mathcal{I}_5 \end{pmatrix} \tag{37}$$

where φ has the same meaning as in Eq. (13). We can again express these in terms of the boundary conditions using Eqs. (37) and (6):

$$K_2 = 2\Omega(x_F - \hat{x}_F) + (\dot{y}_F - \hat{\dot{y}}_F)$$

$$K_3 = \left[\tfrac{3}{2}\Omega(x_F - \hat{x}_F) + (\dot{y}_F - \hat{\dot{y}}_F)\right]\cot(\beta/2) + \tfrac{1}{2}(\dot{x}_F - \hat{\dot{x}}_F)$$

$$K_4 = \tfrac{3}{2}\Omega(x_F - \hat{x}_F) + (\dot{y}_F - \hat{\dot{y}}_F) - \tfrac{1}{2}(\dot{x}_F - \hat{\dot{x}}_F)\cot(\beta/2)$$

$$K_5 = -2\Omega(x_F - \hat{x}_F) + (4/3\beta)(\dot{x}_F - \hat{\dot{x}}_F) - (2\Omega/3\beta)(y_F - \hat{y}_F) - (\dot{y}_F - \hat{\dot{y}}_F) \tag{38}$$

Evaluating these expressions in terms of the Fourier coefficients for the accelerations $T_x(t)$ and $T_y(t)$ gives

$$K_2 = (a_{y0}/2\Omega)\beta, \qquad K_3 = S_{yA}/\Omega + \Omega_F S_{xB}$$

$$K_4 = 2\Omega_F S_{yB} + (1/2\Omega)S_{xA}, \qquad K_5 = S_N \tag{39}$$

where S_{kA} is the same as Eq. (20) and S_{kB} the same as Eq. (21), replacing the Fourier coefficients from the appropriate series expansion of either $T_x(t)$ or $T_y(t)$, and

$$S_N = \frac{\beta}{\pi\Omega}\sum_{n=1}^{\infty}\frac{b_{yn}}{n} \tag{40}$$

The cost function minimizes the thrust in both x and y directions subject to the boundary constraints:

$$J_{xy} = \frac{\beta}{2\Omega}\left(\frac{a_{y0}^2}{2} + \sum_{n=1}^{\infty} a_{yn}^2 + b_{yn}^2\right) + \frac{\beta}{2\Omega}\left(\frac{a_{x0}^2}{2} + \sum_{n=1}^{\infty} a_{xn}^2 + b_{xn}^2\right) + \lambda_2\left(K_2 - \frac{a_{y0}}{2\Omega}\beta\right) + \lambda_3\left(K_3 - \frac{S_{yA}}{\Omega} - \Omega_F S_{xB}\right) + \lambda_4\left(K_4 - 2\Omega_F S_{yB} + \frac{1}{2\Omega}S_{xA}\right) + \lambda_5(K_5 - S_N) \tag{41}$$

Then differentiating with respect to all of the Fourier coefficients, we can find the optimal thrust accelerations in both radial and in-track directions. Once again we can use the same results presented in the Appendix, along with Eq. (A13), to sum over terms in the expansions to get closed-form solutions. The relationship between the Lagrange multipliers and the constraints K_i are now

$$\lambda_2 = \frac{2}{\beta}(\Omega K_2 - \lambda_3)$$

$$\lambda_3 = \Omega\left[\frac{\beta}{4}\left(\frac{\beta + \sin\beta}{1-\cos\beta}\right) + \frac{\beta}{16}\left(\frac{\beta - \sin\beta}{1-\cos\beta}\right) - 1\right]^{-1}\left(\frac{\beta K_3}{2} - K_2\right)$$

$$\lambda_4 = \frac{1}{D}\left[\frac{1}{3\Omega}\left(\frac{\beta}{2} + \frac{8}{3\beta}\right)K_4 - \frac{2}{\beta\Omega}\left(\frac{\beta}{2}\cot\frac{\beta}{2} - \frac{4}{3}\right)K_5\right]$$

$$\lambda_5 = \frac{1}{D}\left[\frac{1}{2\Omega}\left(\frac{\beta - \sin\beta}{1-\cos\beta} + \frac{1}{4}\frac{\beta + \sin\beta}{1-\cos\beta}\right)K_5 - \frac{2}{\beta\Omega}\left(\frac{\beta}{2}\cot\frac{\beta}{2} - \frac{4}{3}\right)K_4\right] \tag{42}$$

where

$$D = \frac{1}{6\Omega^2}\left[\frac{\beta - \sin\beta}{1-\cos\beta} + \frac{1}{4}\frac{\beta + \sin\beta}{1-\cos\beta}\right]\left(\frac{\beta}{2} + \frac{8}{3\beta}\right) - \frac{4}{\beta^2\Omega^2}\left(\frac{\beta}{2}\cot\frac{\beta}{2} - \frac{4}{3}\right)^2 \tag{43}$$

The radial and in-track thrust functions are found by inserting the Fourier coefficients back into the Fourier series. Using Eqs. (A3), (A4), and (A11), we can sum the Fourier series and once again derive closed-form solutions for these components of the thrust acceleration in time:

$$T_x(t) = \tfrac{2}{3}T_1 + (\Lambda/2)\sin(\Omega t - \Phi) \tag{44}$$

$$T_y(t) = T_0 - T_1(\Omega t) + \Lambda\cos(\Omega t - \Phi) \tag{45}$$

where T_0, T_1, and Λ are constants that can most easily be expressed in terms of the Lagrange multipliers:

$$T_0 = \tfrac{1}{2}(\lambda_2 + \lambda_5), \qquad T_1 = \lambda_5/\beta$$

$$\Lambda = \tfrac{1}{2}|\csc(\beta/2)|\left(\lambda_3^2 + \lambda_4^2\right)^{\frac{1}{2}} \tag{46}$$

The thruster phase is given by

$$\Phi = \beta/2 + \upsilon \tag{47}$$

where

$$\lambda_4 = \lambda_3 \tan\upsilon \tag{48}$$

These are the optimal thrust functions for any arbitrary set of initial and final conditions. The cost function can be expressed in the simple form:

$$\mathcal{J}_{xy} = \tfrac{1}{2}\lambda_2 K_2 + \tfrac{1}{2}\lambda_3 K_3 + \tfrac{1}{2}\lambda_4 K_4 + \tfrac{1}{2}\lambda_5 K_5 \tag{49}$$

To find the position and velocity in the orbital plane of the spacecraft during the maneuver, we substitute back into the expressions for I_2 to I_5 to find

$$I_2(t) = T_0 t - \frac{1}{2}\Omega T_1 t^2 + \frac{\Lambda}{\Omega}[\sin(\Omega t - \Phi) + \sin\Phi]$$

$$I_3(t) = \frac{T_0}{\Omega}\sin\Omega t - \frac{4T_1}{3\Omega}(1 - \cos\Omega t) + \frac{5\Lambda t}{8}\cos(\Omega t - \Phi) + \frac{3\Lambda}{16\Omega}[\sin(\Omega t - \Phi) + \sin(\Omega t + \Phi)] \tag{50}$$

$$I_4(t) = \frac{T_0}{\Omega}(1 - \cos\Omega t) - T_1\left(t - \frac{4\sin\Omega t}{3\Omega}\right) + \frac{5\Lambda t}{8}\sin(\Omega t - \Phi) + \frac{3\Lambda}{16\Omega}[\cos(\Omega t - \Phi) - \cos(\Omega t + \Phi)]$$

$$I_5(t) = \frac{1}{2}\Omega t^2\left(T_0 - \frac{2}{3}T_1\Omega t\right) - \frac{4}{9}T_1 t + \Lambda t\sin(\Omega t - \Phi) + \frac{4\Lambda}{3\Omega}[\cos(\Omega t - \Phi) - \cos\Phi] \tag{51}$$

These expressions can then be substituted back into Eq. (6) to obtain the position and velocity of the satellite during the maneuver.

Results

We now consider how we might use these solutions for operating spacecraft in close proximity to each other. We shall consider a default scenario where the origin or guiding center moves in a circular orbit of radius 7000 km about the Earth. The orbital period for such an orbit, which determines Ω is 97.142 min. To illustrate how we might plan maneuvers, we consider two satellites flying in formation, with 200-m separation from the origin both radially and in track, one ahead and one behind. There is also a 10-m cross-track separation with one satellite above the plane and the other below. The satellites have no radial or cross-track velocity, and their in-track velocity relative to the origin is fixed so that $b = 0$, that is, there will be no along-track drift between the satellites and the origin. The formation is required to change so that the two satellites are on the opposite corners of a square of side 400 m on the orbit plane about the origin. The satellites also move to opposite sides of the orbital plane in z, so that the maneuver of one satellite mirrors exactly the movement of the other. The duration of the maneuver can be adjusted to ensure that the maximum thrust required is within the scope of the thrusters and that the satellites remain far enough apart through the maneuver to ensure safety. With these considerations in mind, a transfer time of 11.53 min was chosen for this reorientation of the satellites. In Fig. 1 we show the optimal trajectories derived for this maneuver for each satellite. This is the view on the $z = 0$ plane. The tick marks are at 100-m intervals so that the satellites get only marginally closer than 200 m from each other during the maneuver. The satellite on the right is moving from behind the plane of the page to in front.

In Fig. 2 we show the thrust functions for each of the three thrusters. This is for one satellite, the other being identical but flipped about the horizontal axis. We see that the largest thrust is in the radial direction and corresponds to a little over 4 mN per kg. We have modeled the limits on the thrust accelerations on the Surrey SNAP-1 nanosatellite.[27] The thruster on this small 6.5-kg nanosatellite could

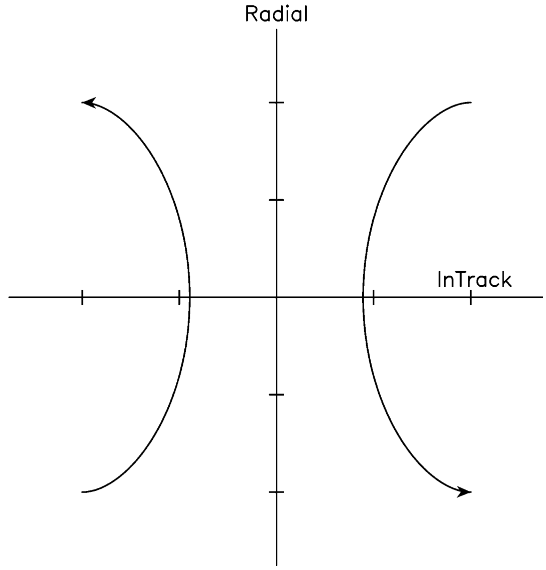

Fig. 1 Earth is towards the bottom, and the mean orbital motion is to the right. The tick marks are at 100-m intervals. Satellite on top right moves from behind the page to in front of it, while the satellite to the left moves from in front to behind.

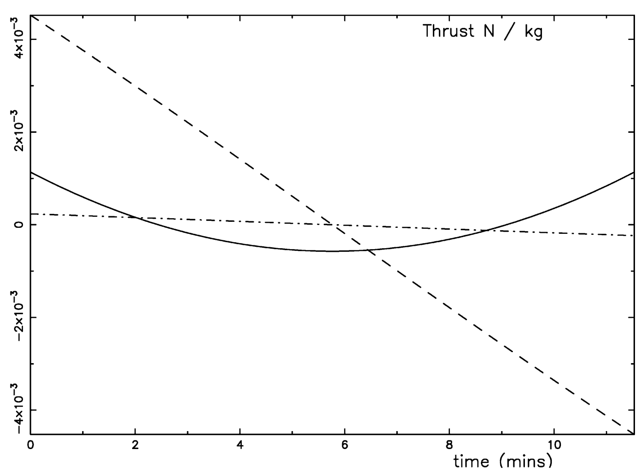

Fig. 2 Thrust accelerations for each of the three thrusters during the maneuver. The along-track thrust is given as a ——, the radial thrust as a - - - -, and the cross-track thrust as a ·—·—·.

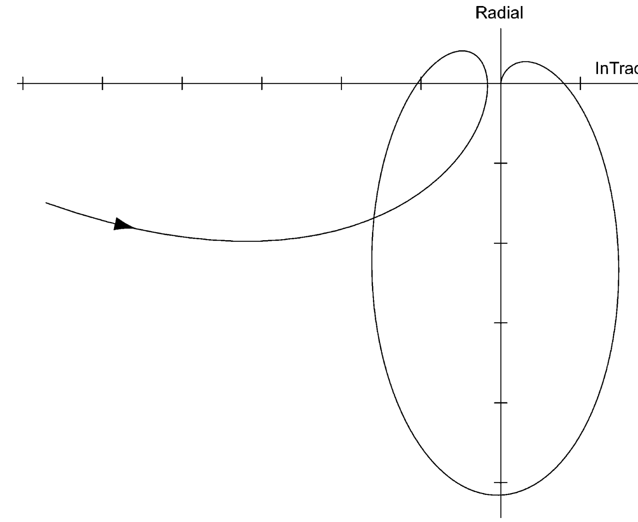

Fig. 3 Orbital approach for a docking satellite that approaches with zero thrust. Tick marks represent 500-m intervals.

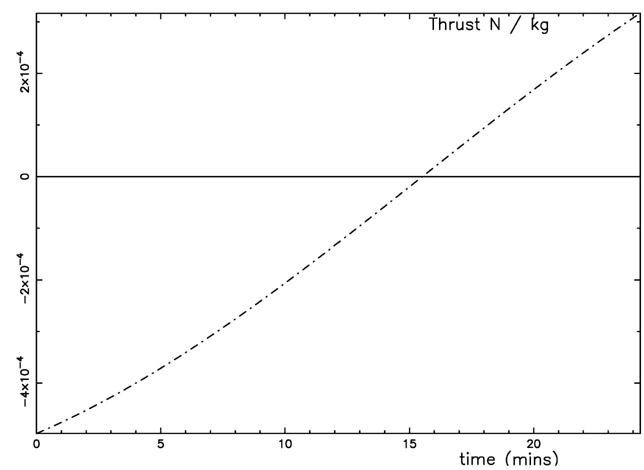

Fig. 4 Thrust function for z thruster in a plane change. The thruster slows the z motion as the satellite approaches the reference orbital plane from below.

deliver 7.7 mN per kg. We see therefore that all of the thrusts required are attainable by even this small satellite.

As a second example of the usefulness of these results, we can consider a docking problem. Here the goal is to reach the origin of the Hill frame with a minimum relative velocity possible. We can use these analytic results to fix the final boundary condition at t_F and the various thrust parameters (T_0, T_1, Λ, Φ, Γ, ζ) and solve backwards to find the initial conditions at an earlier epoch. In this case, β becomes a lookback time rather than a fixed control parameter. To illustrate this, we assume that the approach to docking involved only a single thruster firing in the in-track direction. For a smooth approach to docking, we impose the conditions $T_y(t_F) = \dot{T}_y(t_F) = 0$. These two conditions allow us freedom to choose two of the parameters, say, T_1 and Λ. We can then determine T_0 and Φ from the conditions upon T_y and its derivative:

$$\sin(\beta - \Phi) = -T_1/\Lambda$$

and

$$T_0 = T_1\beta - \Lambda\cos(\beta - \Phi)$$

With the thrust parameters fixed, we can now work back to find the Lagrange multipliers and hence K_2 to K_5. These in turn can be solved for fixed final conditions to determine the initial conditions a time t_F before docking. An example of such a solution is provided in Fig. 3, where we see the approach to the origin on the (x, y) plane. Note how our boundary conditions have imposed a radial approach to docking.

A final example of the usefulness of these formulas is in considering further optimization of the transfer by considering the boundary conditions as free parameters over which we can further optimize. Having the optimal solution as a function of initial and final conditions enables us to minimize the cost with respect to some of these parameters. To illustrate this, consider a cross-track maneuver to try and reduce a difference in inclination between two satellites in a formation. The motion of the second satellite with respect to the first is a z oscillation of amplitude fixed by the inclination difference and period equal to the orbital period. We wish to perform a maneuver to reduce this amplitude of oscillation, but at which phase of the oscillation is it best to perform the maneuver?

It is useful to think in terms of the phase space of $(z, \dot{z})$ for these maneuvers. The trajectory in phase space is a circle centered at the origin, and the radius is determined by the energy of the cross-track excursion. Our maneuver requires us to reduce the energy of this oscillation, and we can maximize the change in energy for a given impulsive change in velocity at the point where the two planes cross (i.e., at $z = 0$).

In Fig. 4 we consider reducing the amplitude of the oscillation from 500 to 300 m. We choose to start the maneuver when the

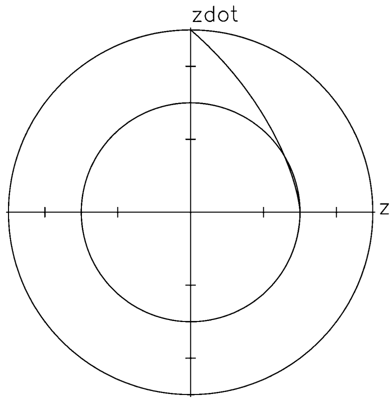

Fig. 5 Trajectory of z oscillation damping as seen in the phase space of the decoupled z motion. Vertical axis has been scaled by Ω so that both axes are measured in meters. The tick marks are at 200-m intervals.

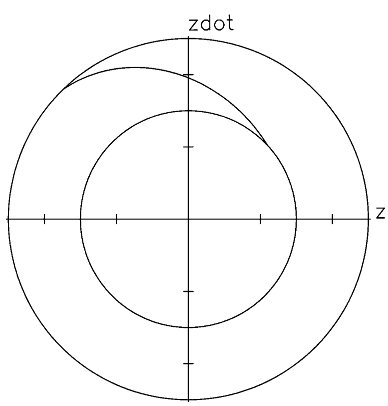

Fig. 6 Optimal transfer by varying oscillation phase at start and end of transfer. Vertical axis has been scaled by Ω so that both axes are measured in meters. The tick marks are at 200-m intervals.

z motion of the satellite is maximum, as it crosses the orbital plane of the reference satellite, and we fix t_F to be a quarter of the orbital period. Figure 4 shows the thrust function obtained for this transfer, and Fig. 5 the trajectory in phase space. Close inspection of this transfer shows that energy in the z oscillation falls below that required for the final oscillation amplitude, that is, the transfer passes inside the inner circle. The inclination is reduced too much, and after 16 min the thrust changes sign to push the two orbital planes further apart. This behavior arises because we have fixed the transfer to a quarter of an orbital period. If we reduced this time by setting $\beta = 1.2$, then we would not have any acceleration phase, and the overall cost reduces by 24.6%.

An alternative way of reducing the cost would be to change the phase of the z oscillation at the required amplitude at which we arrive, that is, the phase around the inner circle in Fig. 5 at which we arrive. Then the natural uncontrolled dynamics will bring the satellite to the correct phase in time. If we treat β as fixed, then we can write conditions (16) and (17) in terms of energy and phase in the respective oscillations. Using Eq. (28) and substituting into Eq. (34) gives the cost function in terms of the phases of the initial and final oscillations. We can then minimise $\mathcal{J}_z$ with respect to these parameters. Doing so shows that the lowest cost transfer for this change in z oscillation amplitude occurs when the initial phase is 134 deg and the final phase is at 43 deg. The transfer is shown in Fig. 6.

The final transfer cost has been reduced by 64% over the initial cost, which shows that significant saving can be made by considering varying the boundary parameters and transfer times. These reductions can be realized in maneuver planning using these analytic solutions and avoid the necessity of exhaustive computations.

Conclusions

In this paper we have considered the arbitrary transfers of a satellite between two relative orbits using the circular Hill's equations. We have considered the situation when the thrusters on the satellite have low levels of thrust, and we are performing maneuvers with a continuous thrust of variable amplitude, but performing attitude maneuvers to change the thrust direction. An optimal control of the maneuver has been studied based on minimizing the propellant usage for the transfer.

A general formulation of the problem has been presented and generic analytic solutions for the thrust functions in each of the three directions of the Hill frame determined. These functions have very simple closed forms, which are parameterized by a number of quantities that we have related to the boundary conditions of the transfer. This provides a simple control law to ensure optimal transfers between arbitrary locations in the Hill frame.

We have considered a number of transfers and illustrated how we can exploit the closed-form solutions to obtain further optimization. We have shown that we can directly put constraints upon the thruster parameters and solve for the boundary conditions we require and the transfer time and how we can treat our cost function as a function of the boundary conditions and exploit natural dynamics to minimize cost if time of transfer is not critical.

The analysis does not take account of mass changes on the satellites as a result of propellant usage, unlike Prussing,[7] which is adequate provided the percentage wet mass of the satellite always remains small. Our analysis also does not take account of the Earth's oblateness. The differential effect of the oblateness leads to a slow relative evolution between satellites, and provided the maneuver time is short compared to the transfer time this differential separation between the satellites can be ignored. In this case, as for the examples chosen, the Hill's frame should be quite adequate. The maneuvers we have considered are all realizable with a small nanosatellite.

Appendix: Some Useful Results

We provide in this Appendix some useful results that we have used in the paper. We start by considering the Fourier series of the functions $\sin\Omega t$ and $\cos\Omega t$, considered over the interval $0 \le t \le t_F$ and considered to be periodically repeated outside this interval. Then

$$\sin\Omega t = \left(\frac{1-\cos\beta}{\beta}\right)\left(1 + 2\Omega^2\sum_{n=1}^{\infty}\frac{\cos n\Omega_F t}{\Omega^2 - n^2\Omega_F^2}\right) + 2\Omega\Omega_F\left(\frac{\sin\beta}{\beta}\right)\sum_{n=1}^{\infty}\frac{n\sin n\Omega_F t}{\Omega^2 - n^2\Omega_F^2} \tag{A1}$$

and

$$\cos\Omega t = \left(\frac{\sin\beta}{\beta}\right)\left(1 + 2\Omega^2\sum_{n=1}^{\infty}\frac{\cos n\Omega_F t}{\Omega^2 - n^2\Omega_F^2}\right) - 2\Omega\Omega_F\left(\frac{1-\cos\beta}{\beta}\right)\sum_{n=1}^{\infty}\frac{n\sin n\Omega_F t}{\Omega^2 - n^2\Omega_F^2} \tag{A2}$$

From these two equations we can derive the relations:

$$1 + 2\Omega^2\sum_{n=1}^{\infty}\frac{\cos n\Omega_F t}{\Omega^2 - n^2\Omega_F^2} = \frac{\beta}{2}\left(\sin\Omega t + \cos\Omega t\frac{\sin\beta}{1-\cos\beta}\right) \tag{A3}$$

and

$$\sum_{n=1}^{\infty}\frac{n\sin n\Omega_F t}{\Omega^2 - n^2\Omega_F^2} = \frac{\beta}{4\Omega\Omega_F}\left(\sin\Omega t\frac{\sin\beta}{1-\cos\beta} - \cos\Omega t\right) \tag{A4}$$

One can also use Parseval's theorem[26] on the series (A1) and (A2) to show that

$$1 - \frac{\sin 2\beta}{2\beta} = 2\left(\frac{1-\cos\beta}{\beta}\right)^2(1 + 2\Omega^4 S) + 4\Omega^2\Omega_F^2\left(\frac{\sin\beta}{\beta}\right)^2 Q \tag{A5}$$

and

$$1 + \frac{\sin 2\beta}{2\beta} = 2\left(\frac{\sin\beta}{\beta}\right)^2 (1 + 2\Omega^4 S) + 4\Omega^2\Omega_F^2 \left(\frac{1-\cos\beta}{\beta}\right)^2 Q \tag{A6}$$

where

$$S = \sum_{n=1}^{\infty} \frac{1}{\left(\Omega^2 - n^2\Omega_F^2\right)^2} \tag{A7}$$

and

$$Q = \sum_{n=1}^{\infty} \frac{n^2}{\left(\Omega^2 - n^2\Omega_F^2\right)^2} \tag{A8}$$

We can solve for S and Q from Eq. (A5) and (A6) provided β is not an integer multiple of 2π or provided $\beta \neq \pi/2 + m\pi$ for integer m. This second restriction is not serious as it analytically continues from values of β on either side of these values.

Solving for these quantities, we can then show that

$$1 + 2\Omega^4 S = \frac{\beta(\beta + \sin\beta)}{4(1-\cos\beta)} \tag{A9}$$

and

$$Q = \frac{\beta(\beta - \sin\beta)}{8\Omega^2\Omega_F^2(1-\cos\beta)} \tag{A10}$$

We next consider a linear function of time: $L(t) = t - t_F/2$ for $0 \leq t \leq t_F$. The Fourier series of this function periodic of period t_F is

$$L(t) = -\frac{t_F}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\Omega_F t}{n} \tag{A11}$$

We will also need to consider the convolution integral between this function and $\sin\Omega t$:

$$\frac{2}{t_F}\int_0^{t_F} L(t_F - t)\sin\Omega t\,\mathrm{d}t = \frac{1}{\Omega}(1+\cos\beta) - \frac{2}{\Omega}\frac{\sin\beta}{\beta} \tag{A12}$$

From this we can show that

$$R = \sum_{n=1}^{\infty}\frac{1}{\Omega^2 - n^2\Omega_F^2} = \frac{1}{2\Omega^2}\left(\frac{\beta}{2}\cot\frac{\beta}{2} - 1\right) \tag{A13}$$

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References

[1] Clohessy, W. H., and Wiltshire, R. S., "Terminal Guidance System for Satellite Rendezvous," *Journal of Aerospace Science*, Vol. 27, No. 5, 1960, pp. 653–658.

[2] Guelman, M., and Aleshin, M., "Optimal Bounded Low-Thrust Rendezvous with Fixed Terminal-Approach Direction," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 2, 2001, pp. 378–385.

[3] Hinz, H. K., "Optimal Low-Thrust near-Circular Orbital Transfer," *AIAA Journal*, Vol. 1, No. 6, 1963, pp. 1367–1371.

[4] Kornhauser, A. L., Lion, P. M., and Hazelrigg, G. A., "An Analytic Solution for Constant-Thrust, Optimal-Coast, Minimum-Propellant Space Trajectories," *AIAA Journal*, Vol. 9, No. 7, 1971, pp. 1234–1239.

[5] Gobetz, F. W., "Optimal Variable-Thrust Transfer of a Power-Limited Rocket Between Neighbouring Circular Orbits," *AIAA Journal*, Vol. 2, No. 2, 1964, pp. 339–343.

[6] Marinescu, A., "Optimal Low-Thrust Orbital Rendezvous," *Journal of Spacecraft and Rockets*, Vol. 13, No. 7, 1976, pp. 385–392.

[7] Prussing, J. E., "Equation for Optimal Power-Limited Spacecraft Trajectories," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 2, 1993, pp. 391–393.

[8] Hall, C. D., and Perez, V., "Minimum-Time Orbital Phasing Manoeuvres," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 6, 2003, pp. 934–941.

[9] Beard, R. W., McLain, T. W., and Hadaegh, F. Y., "Fuel Optimisation for Constrained Rotation of Spacecraft Formations," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 2, 2000, pp. 339–345.

[10] Umehara, H., and McInnes, C. R., "Penalty-Function Guidance for Multiple-Satellite Cluster Formation," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 1, 2005, pp. 182–185.

[11] Van der Ha, J., and Muggellesi, R., "Analytic Models for Relative Motion Under Constant Thrust," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 4, 1990, pp. 644–650.

[12] Vaddi, S. S., Vadali, S. R., and Alfriend, K. T., "Formation Flying: Accommodating Nonlinearity and Eccentricity Perturbations," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 2, 2003, pp. 214–223.

[13] Roger, A. B., and McInnes, C. R., "Safety Constrained Free-Flyer Path Planning at the International Space Station," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 6, 2000, pp. 971–979.

[14] Hablani, H. B., Tapper, M., and Dana-Bashian, D., "Guidance and Relative Navigation for Autonomous Rendezvous in a Circular Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 3, 2002, pp. 553–562.

[15] Moreau, G., and Marcille, H., "RGPS Post-Flight Analysis of the ARP-K Flight," *Proceedings of 11th International Technical Meeting of the Satellite Division of the Institute of Navigation*, Institute of Navigation, Fairfax, VA, Sept. 1998, pp. 1957–1966.

[16] Kawano, I., Mokuno, M., Oda, M., and Anegawa, H., "First Result of Autonomous Rendezvous Docking Technology Experiments on NASDA's ETS-VII Satellite," International Astronautical Congress, IAF-98-A.3.09, Oct. 1998.

[17] Palmer, P. L., and Sweeting, M. N., "Formation Flying—Technology Development at Surrey Space Centre," International Astronautical Congress, Paper IAF-00-A.5.08, Oct. 2000.

[18] Redding, D. C., Adams, N. J., and Kubiak, E. T., "Linear-Quadratic Stationkeeping for the STS Orbiter," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 2, 1989, pp. 248–255.

[19] Ulybyshev, Y., "Long-Term Formation Keeping of Satellite Constellation Using Linear-Quadratic Controller," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 109–115.

[20] Vassar, R. H., and Sherwood, R. B., "Formationkeeping for a Pair of Satellites in a Circular Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 2, 1985, pp. 235–242.

[21] Starin, S. R., Yedavalli, R. K., and Sparks, A. G., "Spacecraft Formation Flying Maneuvers Using Linear-Quadratic Regulation with No Radial Axis Inputs," AIAA Paper 2001-4029, Aug. 2001.

[22] Wenzel, R. S., and Prussing, J. E., "Preliminary Study of Optimal Thrust-Limited Path-Constrained Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 6, 1996, pp. 1303–1309.

[23] Anthony, M. L., and Sasaki, F. T., "Rendezvous Problem for Nearly Circular Orbits," *AIAA Journal*, Vol. 3, No. 9, 1965, pp. 1666–1673.

[24] Ince, E. L., *Ordinary Differential Equations*, Dover, New York, 1956, p. 21.

[25] Stroud, K. A., *Further Engineering Mathematics*, 3rd ed., Palgrave Macmillan, Basingstoke, Hampshire, U.K., 1996, p. 836.

[26] Bracewell, N. R., *The Fourier Transform and Its Applications*, 2nd ed., McGraw–Hill, Tokyo, 1978, p. 413.

[27] Underwood, C. I., Richardson, G., and Savignol, J., "In-Orbit Results from the SNAP-1 Nano-Satellite and Its Future Potential," *Philosophical Transactions of the Royal Society, London. A*, Vol. 361, Dec. 2003, pp. 199–203.